4th Annual Lexington Mathematical Tournament
Team Round

March 30th, 2013

1 Potpourri [70]

1. Alan leaves home when the clock in his cardboard box says 7:35 AM and his watch says 7:41 AM. When he arrives at school, his watch says 7:47 AM and the 7:45 AM bell rings. Assuming the school clock, the watch, and the home clock all go at the same rate, how many minutes behind the school clock is the home clock?

2. Compute
$$\left(\frac{2012^{2012-2013}+2013}{2013}\right)\times 2012.$$
Express your answer as a mixed number.

3. What is the last digit of $2^{3^{4^{5^{6^{7^{8^{9^{\cdot^{\cdot^{\cdot^{2013}}}}}}}}}}}$?

4. Let $f(x)$ be a function such that $f(ab)=f(a)f(b)$ for all positive integers a and b. If $f(2)=3$ and $f(3)=4$, find $f(12)$.

5. Circle X with radius 3 is internally tangent to circle O with radius 9. Two distinct points P_1 and P_2 are chosen on O such that rays $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ are tangent to circle X. What is the length of line segment $\overline{P_1P_2}$?

6. Zerglings were recently discovered to use the same 24-hour cycle that we use. However, instead of making 12-hour analog clocks like humans, Zerglings make 24-hour analog clocks. On these special analog clocks, how many times during 1 Zergling day will the hour and minute hands be exactly opposite each other?

7. Three Small Children would like to split up 9 different flavored Sweet Candies evenly, so that each one of the Small Children gets 3 Sweet Candies. However, three blind mice steal one of the Sweet Candies, so one of the Small Children can only get two pieces. How many fewer ways are there to split up the candies now than there were before, assuming every Sweet Candy is different?

8. Ronny has a piece of paper in the shape of a right triangle ABC, where $\angle ABC = 90°$, $\angle BAC = 30°$, and $AC = 3$. Holding the paper fixed at A, Ronny folds the paper twice such that after the first fold, $\overline{BC}$ coincides with $\overline{AC}$, and after the second fold, C coincides with A. If Ronny initially marked P at the midpoint of $\overline{BC}$, and then marked P' as the end location of P after the two folds, find the length of $\overline{PP'}$ once Ronny unfolds the paper.

9. How many positive integers have the same number of digits when expressed in base 3 as when expressed in base 4?

10. On a 2×4 grid, a bug starts at the top left square and arbitrarily moves north, south, east, or west to an adjacent square that it has not already visited, with an equal probability of moving in any permitted direction. It continues to move in this way until there are no more places for it to go. Find the expected number of squares that it will travel on. Express your answer as a mixed number.

2 Long Answer Section [130]

Write up full solutions on the provided answer sheets. You are allowed to use the results of earlier problems in the section for later ones, even if you have not solved the earlier problems, but not vice versa. This portion of the Team Round consists of three guided problems.

2.1 Fibonacci [35]

Problem: The Fibonacci numbers F_n are defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. It turns out $F_{n+1}^2 - F_{n-1}^2$ is another Fibonacci number. Which one?

1. Calculate $F_{n+1}^2 - F_{n-1}^2$ for $n = 2, 3, 4, 5$. [5]

2. From the pattern observed in Part 1, guess which Fibonacci number is equal to $F_{n+1}^2 - F_{n-1}^2$, in terms of n. (No explanation necessary.) [5]

3. From the definition of F_n, it is true that $F_n = 1(F_{n-1}) + 1(F_{n-2}) = 2(F_{n-2}) + 1(F_{n-3}) = 3(F_{n-2}) + 2(F_{n-3}) = \cdots$. By repeatedly using the definition of the Fibonacci numbers, show that $F_n = F_{n-k}F_{k+1} + F_{n-k-1}F_k$. [15]

4. Directly using the result of Part 3, express the Fibonacci number in Part 2 in terms of three consecutive Fibonacci numbers. [5]

5. Show that the expression for the Fibonacci number that was obtained in Part 4 is equal to $F_{n+1}^2 - F_{n-1}^2$. [5]

2.2 Big Numbers [50]

Problem: Determine whether or not there exist integers $a, b, c, d > 2013$ satisfying the equation $a^2 + b^2 + c^2 + d^2 = abcd + 6$.

6. Suppose b, c, and d are constant and we regard this equation as a quadratic in a. Rewrite the equation in the form $a^2 + ua + v = 0$. What are u and v in terms of b, c, d? [10]

7. What is the sum of the two solutions for a to this equation, in terms of b, c, d? If (a, b, c, d) is a solution to the equation, what is another solution? [15]

8. There is a solution to the equation above satisfying $abcd = 24$. Find this solution. (*Hint: You are not expected to solve algebraically. No explanation is necessary.*) [5]

9. Demonstrate that by starting with the solution found in the previous part and picking the changing variable appropriately at each step, a series of steps can be performed that increases one value in the solution every time. Conclude that a solution does exist with $a, b, c, d > 2013$. [20]

2.3 Hexagon Area [45]

Problem: Let ABC be a triangle and O be its circumcircle. Let A', B', C' be the midpoints of minor arcs AB, BC and CA respectively. Let I be the center of incircle of ABC. If $AB = 13, BC = 14$ and $AC = 15$, what is the area of the hexagon $AA'BB'CC'$?
Suppose $m\angle BAC = \alpha$, $m\angle CBA = \beta$, and $m\angle ACB = \gamma$.

10. Let the incircle of ABC be tangent to $\overline{AB}, \overline{BC}$, and $\overline{AC}$ at J, K, L, respectively. Compute the angles of triangles JKL and $A'B'C'$ in terms of α, β, and γ, and conclude that these two triangles are similar. [10]

11. Show that triangle $AA'C'$ is congruent to triangle $IA'C'$. Show that $AA'BB'CC'$ has twice the area of $A'B'C'$. [15]

12. Let $r = JL/A'C'$ and the area of triangle JKL be S. Using the previous parts, determine the area of hexagon $AA'BB'CC'$ in terms of r and S. [10]

13. Given that the circumradius of triangle ABC is $65/8$ and that $S = 1344/65$, compute r and the exact value of the area of hexagon $AA'BB'CC'$. [10]